Efficient approximate sampling of projection determinantal point processes

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Motivation

Image search/recommendation systems

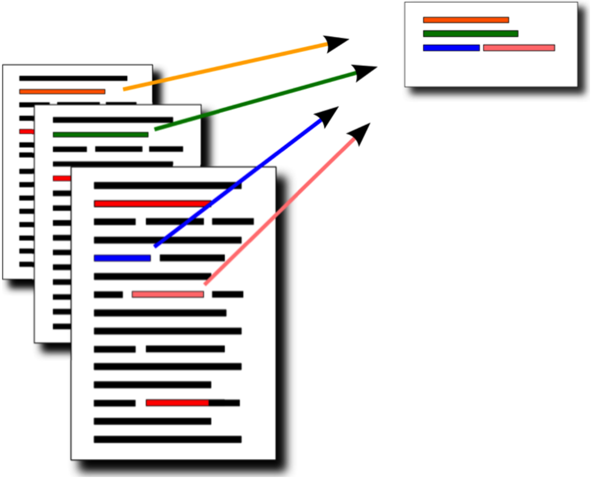
relevance

relevance
+
diversity

'bolt' query

Motivation

Extractive text summarization (Kulesza & Taskar, 2012)



Definition

- $\{1, \ldots, N\}$ indices/labels of items
  - images
  - sentences
  - edges of a graph
- $\mathrm{DPP}(\mathbf{K})$ a measure on subsets of $\{1, \ldots, N\}$
- $\mathbf{K}$ a PSD similarity kernel
- $\mathcal{X} \sim \mathrm{DPP}(\mathbf{K})$ if $\forall S \subseteq \{1, \ldots, N\}$,

$$\mathbb{P}\left[S \subseteq \mathcal{X}\right] = \det \mathbf{K}_S$$

- Existence is guaranteed when $\mathbf{0}_N \preceq \mathbf{K} \preceq \mathbf{I}_N$

Projection DPPs

- $\mathbf{K}$ is an orthogonal projection matrix
  - $\operatorname{Spec} \mathbf{K} \in \{0,1\}^N$

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  - Summaries made of r sentences
  - Bags of r images

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- Gram matrix

$$\mathbf{K} = \sum_{i=1}^{r} u^{(i)} u^{(i)\mathsf{T}} = \mathbf{\Phi}^{\mathsf{T}}\mathbf{\Phi}$$

with $\varphi_n = \left(u_n^{(1)}, \dots, u_n^{(r)}\right)^{\mathsf{T}}$

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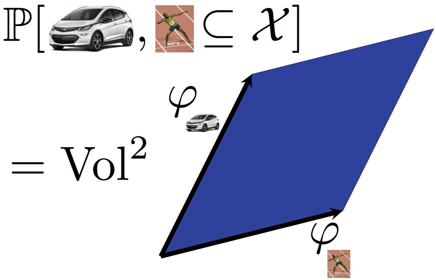
- Geometrical interpretation

$$\mathbb{P}\left[S \subseteq \mathcal{X}\right] = \det \mathbf{K}_S$$
$$= \operatorname{Vol}^2 \{\varphi_i \,;\, i \in S\}$$



Diversity

- Negative association

$$\begin{aligned}\mathbb{P}\left[\{i,j\} \subseteq \mathcal{X}\right] &= \begin{vmatrix}\mathbb{P}\left[i \in \mathcal{X}\right] & \mathbf{K}_{ij} \\ \mathbf{K}_{ij} & \mathbb{P}\left[j \in \mathcal{X}\right]\end{vmatrix} \\ &= \mathbb{P}\left[i \in \mathcal{X}\right]\mathbb{P}\left[j \in \mathcal{X}\right] - \mathbf{K}_{ij}^2 \\ &\leq \mathbb{P}\left[i \in \mathcal{X}\right]\mathbb{P}\left[j \in \mathcal{X}\right]\end{aligned}$$

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- The larger $|\mathbf{K}_{ij}|$ the smaller $\mathbb{P}\left[\{i,j\} \subseteq \mathcal{X}\right]$
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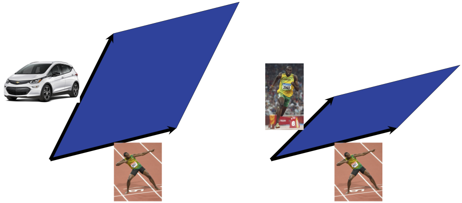
$\mathbb{P}[$  ,  $\subseteq \mathcal{X}] \geq \mathbb{P}[$  ,  $\subseteq \mathcal{X}]$



Setup

- Build the $r \times N$ feature matrix

$$\mathbf{A} = \left(\sqrt{q_1}\phi_1 | \dots | \sqrt{q_N}\phi_N\right)$$

  - If $\|\phi_i\|^2 = 1$, 'angles' encode diversity
  - q_i measures relevance of item i

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- If $\mathcal{X} \sim \mathrm{DPP}(\mathbf{K})$,
  - $|\mathcal{X}| \overset{a.s.}{=} r$
  - For $B = \{i_1, \dots, i_r\}$,

$$\mathbb{P}\left[\mathcal{X} = B\right] \propto \left|\det \mathbf{A}_{:B}\right|^2 = \mathrm{Vol}^2 \left\{\sqrt{q_i}\phi_i \, ; \, i \in B\right\}$$

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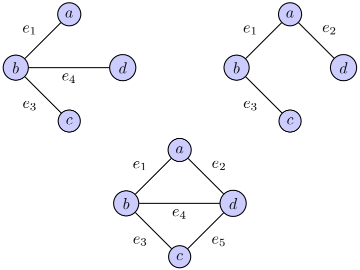
- $\operatorname{DPP}(\mathbf{K})$ has support

$$\mathcal{B} \triangleq \left\{B \,;\, |B| = r, \det \mathbf{A}_{:B} \neq 0\right\}$$

i.e. collection of columns of $\mathbf{A}$ forming a basis

One example

Uniform spanning trees (Lyons, 2003)



Vertex-edge incidence matrix

$$
\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}
\begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\
\left[\begin{array}{ccccc} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 \end{array}\right] \end{array}
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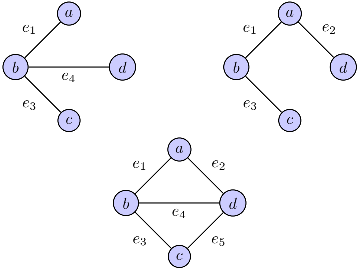

$$\mathbf{A} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_5 \\ -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

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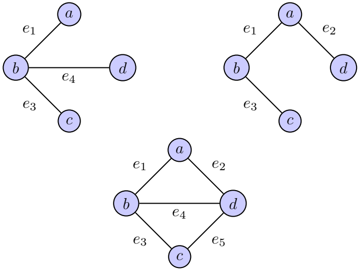

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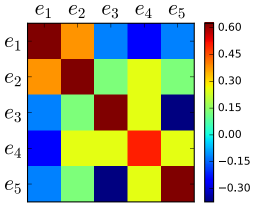

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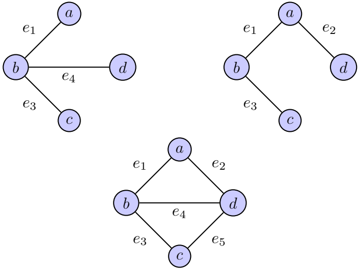

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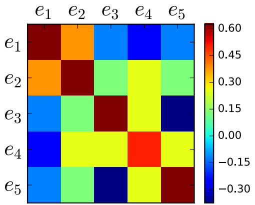

- DPP($\mathbf{K}$) is a measure on the edge set of G
  - $\mathcal{B}$ = collection of spanning trees of G
- $\mathbb{P}\left[\mathcal{X} = B\right] = \dfrac{\left|\det \mathbf{A}_{:B}\right|^2}{\det \mathbf{A}\mathbf{A}^\top} = \dfrac{1}{|\mathcal{B}|} \mathbb{1}_{B \in \mathcal{B}}$

Rk: Unimodularity, transfer current matrix, matrix tree theorem, Laplacian solvers

Exact sampling

- From $r \times N$ feature matrix $\mathbf{A} = \left(\phi_1 | \dots | \phi_N\right)$
- To $N \times N$ projection kernel $\mathbf{K} = \mathbf{A}^\top \left[\mathbf{A}\mathbf{A}^\top\right]^{-1} \mathbf{A}$

Exact sampling (Hough et al., 2006; Kulesza & Taskar, 2012)

Sample $\mathcal{X} \sim \mathrm{DPP}(\mathbf{K})$

- Marginals
$$\mathbb{P}\left[\mathcal{X} = B\right] = \det \mathbf{K}_B$$
- Chain rule, $J = \{i_1, \dots, i_k\}$
$$\mathbb{P}\left[i_{k+1} = i | J\right] \propto \mathbf{K}_{ii} - \mathbf{K}_{i,J} {\mathbf{K}_J}^{-1} \mathbf{K}_{J,i}$$
- Costly: eigen-decomposition + Gram-Schmidt = $\mathcal{O}(N^3 + Nr^2)$

Rk: Uniform spanning trees (Aldous, 1990; Propp & Wilson, 1998), generation of mazes

Approximate sampling - 1

- From $r \times N$ feature matrix $\mathbf{A} = \left(\phi_1 | \dots | \phi_N\right)$

Approximate sampling (Anari et al., 2016; Li et al., 2016)

Build a Markov chain, $\mathbf{B} \triangleq \mathbf{A}_{:B}$

- State space $\mathcal{B} \triangleq \{B\,;\, \det \mathbf{B} \neq 0\}$
- Stationary distribution

$$\propto |\det \mathbf{B}|^2 = \mathrm{Vol}^2 \{\phi_i\,;\, i \in B\} \cdot \mathbb{1}_{B \in \mathcal{B}}$$

- Basis-exchange graph
  - $B \leftrightarrow B' = (B \setminus \{i\}) \cup \{j\}$
  - Full analysis: polynomial mixing time
  - Local and correlated moves on $\mathcal{B}$

Rk: Basis-exchange graph (Feder & Mihail, 1992), matroid theory (Oxley, 2003)

Approximate sampling - 1

But a dream within a dream?
Is all that we see or seem
One from the pitiless wave?
O God! can I not save
Them with a tighter clasp?
O God! can I not grasp
While I weep--while I weep!
Through my fingers to the deep,
How few! yet how they creep
Grains of the golden sand-
And I hold within my hand
Of a surf-tormented shore,
I stand amid the roar
Is but a dream within a dream.
All that we see or seem
Is it therefore the less gone?
In a vision, or in none,
In a night, or in a day,
Yet if hope has flown away
That my days have been a dream;
You are not wrong, who deem
Thus much let me avow-
And, in parting from you now,
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Approximate sampling - 2

- From $r \times N$ feature matrix $\mathbf{A} = \left(\phi_1 | \dots | \phi_N\right)$

Approximate sampling (G., Bardenet & Valko, 2017)

Build a Markov chain, $\mathbf{B} \triangleq \mathbf{A}_{:B}$

- State space $\mathcal{B} \triangleq \{B\,;\, \det \mathbf{B} \neq 0\}$
- Stationary distribution

$$\propto \left|\det \mathbf{B}\right|^2 = \mathrm{Vol}^2 \{\phi_i\,;\, i \in B\} \cdot \mathbb{1}_{B \in \mathcal{B}}$$

- Wander in a continuous embedding of $\mathcal{B}$
  - Geometrical representation of $\mathcal{B}$
  - More decorelated moves, empirically faster mixing

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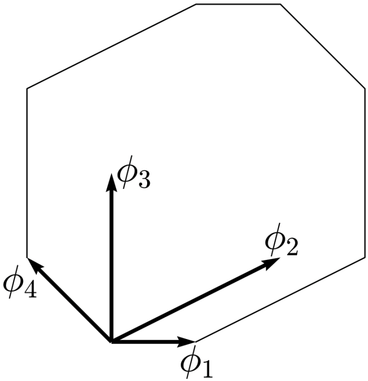
Continuous embedding of the state space $\mathcal{B}$

Volume spanned by feature vectors

$$\mathcal{Z}(\mathbf{A}) \triangleq \mathbf{A}[0,1]^N$$



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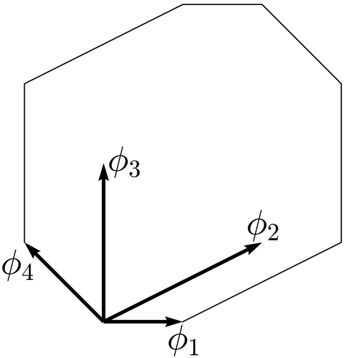
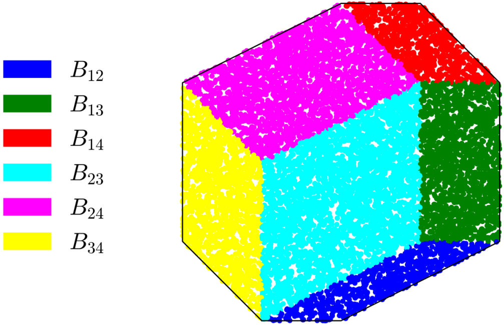




admits a natural tiling (Dyer & Frieze, 1994), $\mathbf{B} \triangleq \mathbf{A}_{:B}$

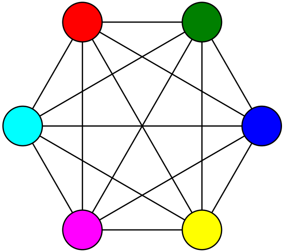
$$\mathrm{Vol}\,\mathcal{Z}(\mathbf{A}) = \sum_{B\in\mathcal{B}} \mathrm{Vol}\,\mathbf{B} = \sum_{B\in\mathcal{B}} |\det \mathbf{B}|$$

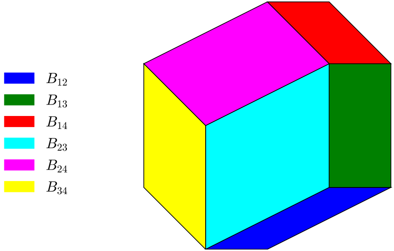
Random walk on $\mathcal{B}$ i.e. on tiles

- From $r \times N$ feature matrix $\mathbf{A} = \left(\phi_1 | \ldots | \phi_N\right)$
- Limiting distribution, $\mathbf{B} \triangleq \mathbf{A}_{:B}$

$$\mathbb{P}\left[\mathcal{X} = B\right] \propto \mathrm{Vol}^2 \, \mathbf{B} \cdot \mathbb{1}_{B \in \mathcal{B}}$$

- State space $\mathcal{B} \triangleq \{B\,;\, \det \mathbf{B} \neq 0\}$
- Continuous embedding of $\mathcal{B}$ via tiling of $\mathcal{Z}(\mathbf{A}) = \mathbf{A}[0,1]^N$

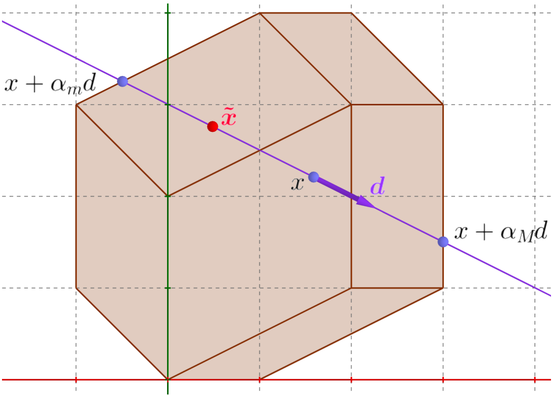




Random walk on $\mathcal{B}$ i.e. on tiles

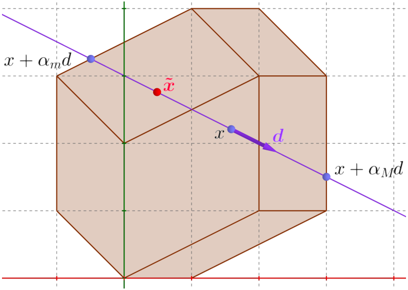
Underlying continuous walk

- $\mathcal{Z}(\mathbf{A})$ is a polytope (convex)
- Hit-and-run is efficient for convex bodies (Lovász & Vempala, 2003)



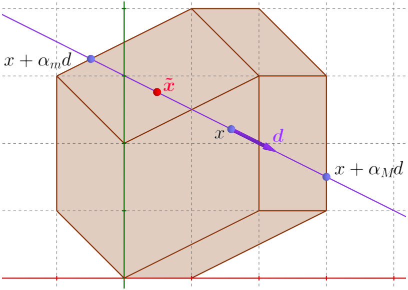
Random walk on $\mathcal{B}$ i.e. on tiles

Continuous random walk on $\mathcal{Z}(\mathbf{A})$



Random walk on $\mathcal{B}$ i.e. on tiles

Continuous random walk on $\mathcal{Z}(\mathbf{A})$



Discrete random walk on $\mathcal{B}$

- Identify the tile in which x lies

$$\begin{aligned} \min_{y \in \mathbb{R}^N} \quad & c^\top y \\ \text{s.t.} \quad & \mathbf{A} y = x \\ & 0 \leq y \leq 1 \end{aligned}$$

- $B_x = \{i\,;\, y_i^* \in]0,1[\}$

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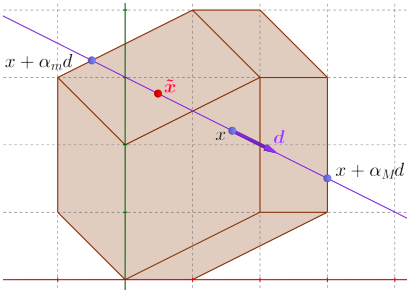

Continuous target distribution

$$\pi(x)\,\mathrm{d}x = \sum_{B \in \mathcal{B}} C_B \times \mathbb{1}_{\mathbf{B}}(x)\,\mathrm{d}x$$

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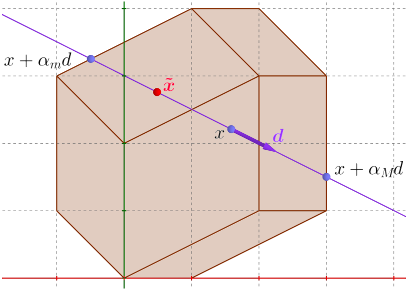

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$$\mathbb{P}\left[B_x = B\right] \propto \int_{\mathbf{B}} \pi(x)\,\mathrm{d}x = C_B \times \operatorname{Vol} \mathbf{B}$$

Acceptance = 1

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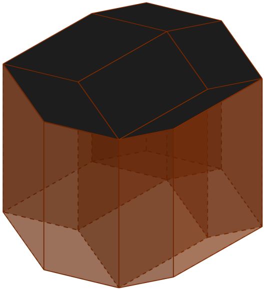

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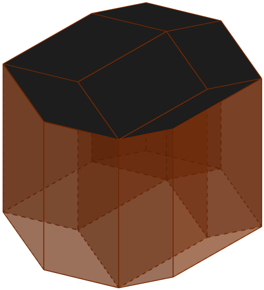

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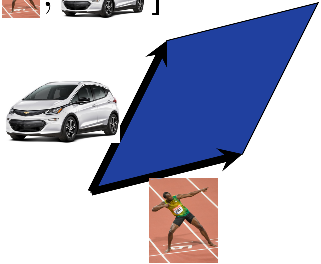
$$\mathbb{P}\left[B_x = B\right] \propto 1 \times \mathrm{Vol}\,\mathbf{B} = \mathrm{Vol}^{1}\,\mathbf{B}$$

$\mathbb{P}[B_x =$ ,  $]$

$\propto \mathrm{Vol}^{1}$



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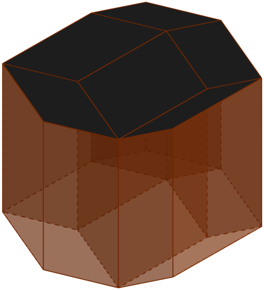

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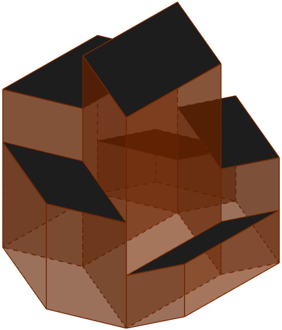

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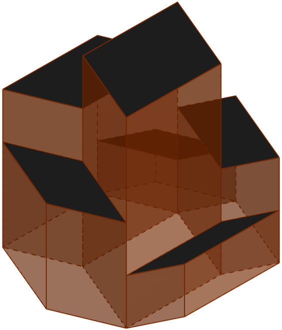

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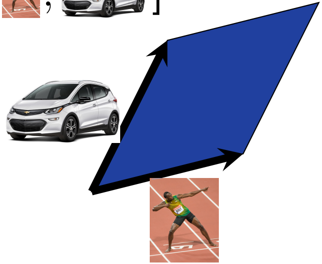

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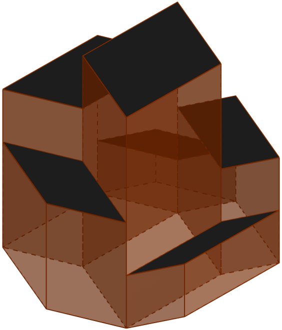

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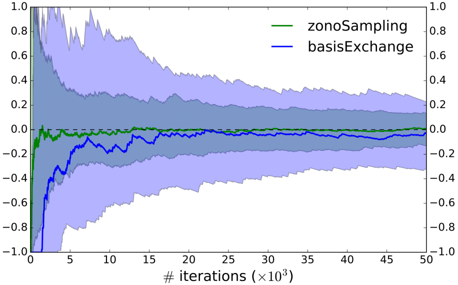
Relative error of the estimation of $\mathbb{P}\left[\{i_1, i_2, i_3\} \subseteq \mathcal{X}\right] = \det \mathbf{K}_{\{i_1,i_2,i_3\}}$



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Fast sampling of projection DPPs?

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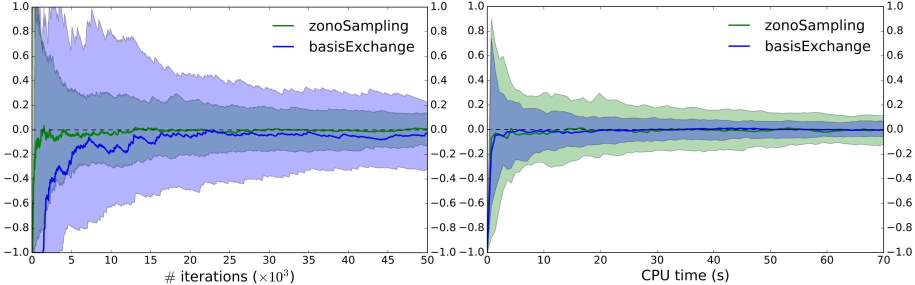

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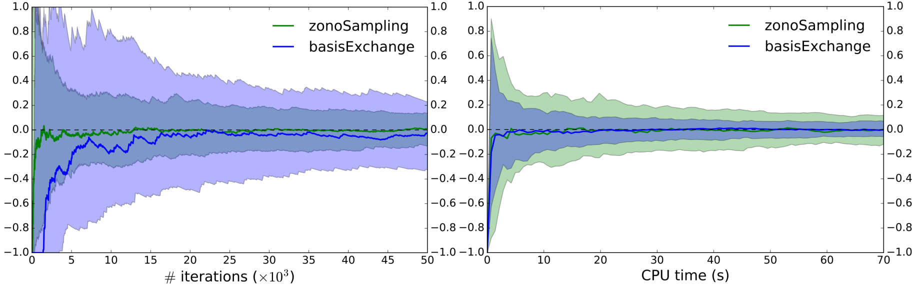

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Efficient sampling of projection DPPs!

Summarizing a news article from Slate

Find Y to maximize (Kulesza & Taskar, 2012)

$$\int \textsc{Rouge-1F}(Y, Z)\,\mathrm{DPP}(Z)dZ \approx \frac{1}{N}\sum_{i=1}^{N} \textsc{Rouge-1F}\,(Y, Y_i)$$

where Y_i are samples from our Markov chain

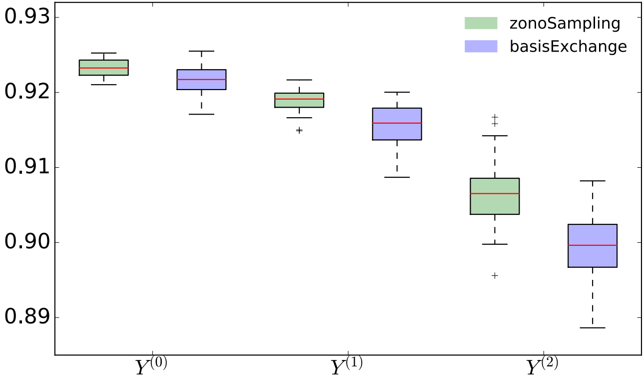

Figure 1: Estimation of the integrated cost

Conclusion

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  - Build DPP($\mathbf{A}^{\top}(\mathbf{A}\mathbf{A}^{\top})^{-1}\mathbf{A}$)
  - Continuous embedding of the state space
  - New bridge MCMC $\cap$ Optimization $=$ hit-and-run $+$ LPs
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POSTER #80

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