Sampling projection DPPs

Sampling \equiv sequential Gram-Schmidt orthogonalization of feature vectors \( \Phi(x_1), \ldots, \Phi(x_N) \)
where \( K(x,y) = \Phi(x)^T \Phi(y) \), with \( \Phi(x) = (\phi_0(x), \ldots, \phi_{N-1}(x)) \).

Apply the chain rule to sample \( \{x_1, \ldots, x_N\} \) and forget the order the points were selected

\[
(1) = \frac{\|\Phi(x_1)\|^2}{N} \omega(x_1) \prod_{n=2}^{N} \text{distance}^2(\Phi(x_n), \text{span}\{\Phi(x_{n-1})\}) \omega(x_n) \ dx_n
\]

Efficient sampler for a specific DPP

\textbf{Sampling the Jacobi ensemble}

- \( d = 1 \): eignals of a random \( (m \times m) \) matrix
- \( d \geq 2 \): chain rule (2) with rejection sampling
  
  \[
  x_1 \sim \text{proposal } \omega_1(x) \ dx \\
  \text{rejection constant } \leq 2^d
  \]

Total number of rejections \( \approx 2^d N \log(N) \)

\textbf{First estimator (BH, 2019)}

\[
\hat{I}_N^{BH}(f) \triangleq \frac{1}{N} \sum_{n=1}^{N} \frac{1}{K(x_n, x_n)} f(x_n)
\]

Interpretable \( \omega(x_n) \equiv \text{random Gaussian quadrature} \)

\textbf{Second estimator (EZ, 1960)}

Let \( f = \sum_{i=0}^{M-1} (f, \phi_i) \phi_i, \quad 1 \leq M \leq \infty \)
Take \( \{x_1, \ldots, x_N\} \sim \text{DPP}(\mu, \sum_{i=0}^{M-1} \phi_i \phi_i(y)) \)

Solve the linear system

\[
\begin{pmatrix}
(\phi_0(x_1) \ldots \phi_{N-1}(x_1)) & y_1 \\
(\phi_0(x_N) \ldots \phi_{N-1}(x_N)) & y_N
\end{pmatrix}
= \left( \begin{array}{c} f(x_1) \\ f(x_N) \end{array} \right)
\]

Get unbiased estim\(\hat{\text{ests}}\) of "Fourier coeff"s

- \( \mathbb{E}[y_i] = (f, \phi_i) \)

with interpretable \& practical variance

- \( \text{Var}[y_i] = \|f\|^2 - \sum_{l=0}^{M-1} (f, \phi_l)^2 \) if \( N \geq M \)

- \( \text{Cov}[y_i, y_j] = 0, i \neq j \)

When \( \phi_0 \) is constant, e.g., Jacobi ensemble

\[
\hat{I}_N^{EZ}(f) \triangleq \frac{M}{\phi_0} \mu^{1/2} \frac{M!}{M^{M+1}} \text{det} \Phi \text{det} f
\]

Non obvious \( \omega_1(x_1, \ldots, x_N) \leq 0 \sum_{n=0}^{\infty} \omega_n = \mu(X) \)

\[
\mathbb{E} [I_N^{EZ}] = \int f(x)^2 \mu(\text{dx}) \text{ unbiased}
\]

\[
\text{Var} [I_N^{EZ}] = \int \|f\|^2 - \sum_{l=0}^{M-1} (f, \phi_l)^2 \text{ if } N \geq M
\]

with the Jacobi ensemble

\[
\hat{I}_N^{EZ}(f) \triangleq \frac{M}{\phi_0} \mu^{1/2} \frac{M!}{M^{M+1}} \text{det} \Phi \text{det} f
\]

Experiments

\[
f = \text{smooth bump function, } d = 1, 2, 3, 4
\]

\[
f = \sum_{i=0}^{M-1} (f, \phi_i) \phi_i, \quad M = 70, \quad d = 2, 4
\]

\[
f = \sum_{i=0}^{M-1} (f, \phi_i) \phi_i, \quad N = N + 1, \quad d = 2, 4
\]

Take home message

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punchlines contributions

Jacobi ensemble

Guillaume Gautier, Rémi Bardenet, and Michal Valko

On two ways to use Determinantal Point Processes for Monte Carlo integration

